Pergamon

Int. J. Solids Structures Vol. 34, Nos 33–34, pp. 4263–4280, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(97)00024–3

DECAY RATE OF SAINT-VENANT END EFFECTS FOR MULTILAYERED ORTHOTROPIC STRIPS

NERIO TULLINI and MARCO SAVOIA
University of Bologna, Faculty of Engineering, DISTART-Structural Engineering, Viale Risorgimento 2, 40136 Bologna, Italy

(*Received* 20 *March* 1996; *in revised form* 8 *January* 1997)

Abstract—Saint-Venant decay rate of end effects is investigated for generally laminated orthotropic strips under self-equilibrated end loads. The problem is governed by a fourth-order partial differential equation for the Airy stress function with discontinuous coefficients at the layer interfaces, where displacements and traction continuities are imposed. The solution is found in the form of product of an exponentially decaying function along the strip and an unknown function defined over its total height. External face and interface conditions are used to obtain the characteristic equation for the eigenvalues governing the decay rate of end effects along the strip. For orthotropic and strongly orthotropic sandwich strips the transcendental eigenvalue equation is explicitly given. The case of laminated strips with periodic layout is finally considered. Making use of the homogenization method, both effective elastic constants and expressions for the local stress variation at the layer level are obtained. Numerical calculations confirm that the eigenvalues of the homogenized material are the asymptotic values of those of periodically laminated strips when the number of layers increases. Moreover, homogenization method is shown to be very powerful also in the calculation of local stress distributions. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The analysis of local effects for beams, plates or shells is a classical problem of structural mechanics. Often Saint-Venant's principle is invoked to separate the region (interior domain) where the solution depends on the load resultants only, and that close to the loaded region where a more complex analysis is required.

With reference to composite materials, many approaches have been proposed to obtain detailed descriptions of stresses near loaded regions as well as to evaluate the decay length of end effects. In fact, it is well established that local effects are very important for strongly anisotropic or multilayered structural elements, since their decay is typically much slower than for isotropic materials (Horgan, 1982, 1989, 1996; Horgan and Knowles, 1983; Horgan and Simmonds, 1994). This phenomenon has important practical consequences in many fields. In the mechanical testing of anisotropic materials, appropriate specimen sizes and strain gauge placements must be adopted to disregard the effects of clamping of the extremities (Folkes and Arridge, 1975). Local stresses are also very important for composite material lap joints and plating systems by means of steel plates or thin Fiber Reinforced Plastic laminae (Plevris and Triantafillou, 1992). Moreover, the knowledge of decay length of end effects or local loadings is important in order to appropriately select FEM meshes for numerical solutions concerning complex structures.

The simplest problem where end effects can be investigated is the semi-infinite strip subject to self-equilibrated load distributions. For homogeneous isotropic strips, the classical Fadle-Papkovich eigenfunctions have been obtained (Timoshenko and Goodier, 1970); the exponential decay rate of end effects is characterized by the eigenvalue with smallest positive real part. This solution has been extended in Choi and Horgan (1977), Crafter *et al.* (1993) and Wang *et al.* (1993) to cover the cases of orthotropic and anisotropic homogenous materials. For strongly orthotropic strips (typically fiber-reinforced composites), the characteristic decay length has been found to be of the order of $H(E_1/G_{12})^{1/2}$, where H, E_1 and G_{12} are the strip height, longitudinal Young modulus and shear modulus, so assessing the importance of end effects. In this case, both eigenvalue conditions and solutions of full boundary-value problems can be simplified considerably. In fact, biorthogonality conditions between the exact complex eigenfunctions reduce to simple (and more tractable) orthogonality relations between real eigenfunctions (Horgan and Simmonds, 1991; Savoia and Tullini, 1996).

Analytical results for isotropic sandwich strips were first obtained by Choi and Horgan (1978) by solving the biharmonic equation for the Airy stress function with perfect bonding at the interfaces and traction free lateral sides. They showed that very slow stress decay arises when the Young's modulus of the core is small compared with that of the external faces. Anti-plane shear deformations of symmetric sandwich structures with anisotropic layers have been examined recently by Baxter and Horgan (1995).

Few analytical studies have been carried out for laminated strips. Wijeyewickrema (1995) analyzed a very particular symmetric deformation of a multilayered composite consisting of two dissimilar alternating layers of isotropic materials, subject to self-equilibrated loads symmetric with respect to each layer mid-plane, so that the representative domain reduces to a sandwich cell with null transverse displacement and null shear stresses at the external faces. The exponential decay rate has been written in terms of the two Dundurs' constants (1969). Multilayered strips have been studied by Dong and Goetschel (1982) making use of a finite element interpolation over the laminate height.

In the present paper, an analytical procedure to obtain the eigenvalues and corresponding eigenfunctions for a generally laminated orthotropic strip is presented. The Airy stress function is taken as the product of an exponentially decaying function in the axial direction and an unknown function (the eigenfunction) over the strip height. For orthotropic sandwich strips, the characteristic eigenvalue equation is explicitly given for both symmetric and antisymmetric deformations. It is shown that this equation depends on five combinations of elastic constants. Approximate eigenvalue equations are also obtained for strongly orthotropic sandwich strips, depending on two elastic parameters only. For typical composite sandwich strips the approximate eigenconditions usually give rise to small errors with respect to the exact elasticity solution.

Finally, a homogenization technique is used to obtain effective elastic moduli and expressions for local stresses at the layer level of a laminated strip with periodic layout. The numerical applications confirm that the eigenvalues computed for the homogenized material represent the asymptotic values when the number of layers increases, for both symmetric and asymmetric laminations. An upper bound for the relative error of the homogenized eigenvalues with respect to the exact values has been found to be of $O(n_c^{-2})$, where n_c is the number of elementary cells of periodicity, independently of the stacking sequence and layer material properties considered.

In order to confirm the importance of these studies, it should be remembered that eigenfunction expansions can be used to solve boundary-value problems for multilayered strips subject to prescribed self-equilibrated end load distributions or displacements. From the technical point of view, the interest in this kind of analysis for laminated composites is that local effects represent the most relevant cause of delamination failures. In order to carry out a boundary-value analysis, the completeness of eigenfunctions sets must be proved. To date, this problem has been solved for the Fadle-Papkovich eigenfunctions for homogeneous isotropic strips only (Gregory, 1980). For the problem at hand, the eigenvalue problem is not self-adjoint, and bi-orthogonality relations between eigenfunctions must be derived, as has been done in Choi and Horgan (1977) for orthotropic strips and, in a more general case, in Gregory (1983). An alternative approach for boundary layer analysis based on asymptotic expansion is proposed in Dumontet (1986).

2. GOVERNING EQUATIONS

Consider a semi-infinite multilayered rectangular strip of total height $H = 2h$. Reference axes x_1 and x_2 are set in the axial and transverse directions, respectively, so that the strip domain is $x_1 \geqslant 0$, $-h \leqslant x_2 \leqslant h$. The strip is made of S orthotropic, linearly elastic layers of height $h^{(s)}$, with orthotropy axes coinciding with the reference axes. The long faces of the strip ($x_2 = \pm h$) are traction free and the end section $x_1 = 0$ is subject to self-equilibrated load distributions.

The stress components $\sigma_{\alpha\beta}^{(s)}$ $(\alpha, \beta = 1, 2)$ of the sth layer, satisfying the equilibrium equations with null body forces, are written in terms of the classical Airy stress function $F^{(s)}(x, y)$:

$$\sigma_{11}^{(s)} = F_{,22}^{(s)}, \quad \sigma_{22}^{(s)} = F_{,11}^{(s)}, \quad \sigma_{12}^{(s)} = -F_{,12}^{(s)} \tag{1}$$

where a subscript preceded by a comma denotes partial differentiation. The compatibility equation yields the governing differential equation (Lekhnitskii, 1981):

$$F_{,2222}^{(s)} + \bar{E}^{(s)} F_{,1122}^{(s)} + (\varepsilon^{(s)} \bar{E}^{(s)})^2 F_{,1111}^{(s)} = 0 \tag{2}$$

where

$$\bar{E}^{(s)} = \frac{R_{66}^{(s)} + 2R_{12}^{(s)}}{R_{11}^{(s)}}, \quad (\varepsilon^{(s)} \bar{E}^{(s)})^2 = \frac{R_{22}^{(s)}}{R_{11}^{(s)}} \tag{3}$$

and the constants $R_{ij}^{(s)}$ are the usual reduced elastic coefficients of the sth layer. The non-dimensional parameter defined in (3b) is identical to ε introduced by Miller and Horgan (1995a, b). The coefficients of eqns (3) are given by

$$R_{11}^{(s)} = \frac{1}{E_1^{(s)}}, \quad R_{22}^{(s)} = \frac{1}{E_2^{(s)}}, \quad R_{12}^{(s)} = -\frac{\nu_{12}^{(s)}}{E_1^{(s)}}, \quad R_{66}^{(s)} = \frac{1}{G_{12}^{(s)}} \tag{4}$$

$$\bar{E}^{(s)} = \frac{E_1^{(s)}}{G_{12}^{(s)}} - 2\nu_{12}^{(s)}, \quad (\varepsilon^{(s)} \bar{E}^{(s)})^2 = \frac{E_1^{(s)}}{E_2^{(s)}} \tag{5}$$

for generalized plane stress or

$$R_{11}^{(s)} = \frac{1 - \nu_{13}^{(s)} \nu_{31}^{(s)}}{E_1^{(s)}}, \quad R_{22}^{(s)} = \frac{1 - \nu_{23}^{(s)} \nu_{32}^{(s)}}{E_2^{(s)}}, \quad R_{12}^{(s)} = -\frac{\nu_{12}^{(s)} + \nu_{13}^{(s)} \nu_{32}^{(s)}}{E_1^{(s)}}, \quad R_{66}^{(s)} = \frac{1}{G_{12}^{(s)}} \tag{6}$$

$$\bar{E}^{(s)} = \frac{1}{1 - \nu_{13}^{(s)} \nu_{31}^{(s)}} \left[\frac{E_1^{(s)}}{G_{12}^{(s)}} - 2(\nu_{12}^{(s)} + \nu_{13}^{(s)} \nu_{32}^{(s)}) \right], \quad (\varepsilon^{(s)} \bar{E}^{(s)})^2 = \frac{E_1^{(s)} (1 - \nu_{23}^{(s)} \nu_{32}^{(s)})}{E_2^{(s)} (1 - \nu_{13}^{(s)} \nu_{31}^{(s)})} \tag{7}$$

for plane strain, where $E_\alpha^{(s)}$, $G_{12}^{(s)}$, $\nu_{ij}^{(s)}$ denote Young's moduli, shear modulus and Poisson ratio of the sth layer, respectively.

Introducing the dimensionless variables $x = x_1/h$, $y = x_2/h$, the strip domain reduces to $x \geqslant 0$, $|y| \leqslant 1$. In the spirit of Saint-Venant's principle, a decaying solution of eqn (2) with boundary conditions representing self-equilibrated load distributions is sought in the form:

$$F^{(s)}(x, y) = h^2 e^{-\lambda x} \psi^{(s)}(y), \tag{8}$$

where λ is the eigenvalue, which may be either real or complex, characterising the decay rate of $F(x, y)$ along x. The stress components (1) for the sth layer may then be written as

$$\sigma_{11}^{(s)} = e^{-\lambda x} \frac{d^2 \psi^{(s)}}{dy^2}, \quad \sigma_{22}^{(s)} = \lambda^2 e^{-\lambda x} \psi^{(s)}, \quad \sigma_{12}^{(s)} = \lambda e^{-\lambda x} \frac{d\psi^{(s)}}{dy}. \tag{9}$$

Substituting eqn (8) into eqn (2) yields a fourth-order ordinary differential equation for the eigenfunction $\psi^{(s)}(y)$ (Choi and Horgan, 1977); for orthotropic layers ($\varepsilon \neq 1/2$) the corresponding solution is:

$$\psi^{(s)}(y) = A_1^{(s)} \cos\frac{c_1^{(s)}+c_2^{(s)}}{2}\lambda y + A_2^{(s)} \sin\frac{c_1^{(s)}+c_2^{(s)}}{2}\lambda y + A_3^{(s)} \cos\frac{c_1^{(s)}-c_2^{(s)}}{2}\lambda y + A_4^{(s)} \sin\frac{c_1^{(s)}-c_2^{(s)}}{2}\lambda y \qquad (10)$$

where $A_i^{(s)}$ are integration constants (for each layer) and

$$c_1^{(s)} = \sqrt{\bar{E}^{(s)}(1+2\varepsilon^{(s)})}, \quad c_2^{(s)} = \sqrt{\bar{E}^{(s)}(1-2\varepsilon^{(s)})}. \qquad (11)$$

The eigenfunction (10) may be alternatively rewritten in the form:

$$\psi^{(s)}(y) = \bar{A}_1^{(s)} \cos\frac{c_1^{(s)}\lambda y}{2}\cos\frac{c_2^{(s)}\lambda y}{2} + \bar{A}_2^{(s)} \sin\frac{c_1^{(s)}\lambda y}{2}\sin\frac{c_2^{(s)}\lambda y}{2} + \bar{A}_3^{(s)} \sin\frac{c_1^{(s)}\lambda y}{2}\cos\frac{c_2^{(s)}\lambda y}{2} + \bar{A}_4^{(s)} \cos\frac{c_1^{(s)}\lambda y}{2}\sin\frac{c_2^{(s)}\lambda y}{2}. \qquad (12)$$

For an isotropic layer, eqns (3) give $\bar{E}^{(s)} = 2$, $\varepsilon^{(s)} = 1/2$, and eqn (2) reduces to the classical biharmonic equation. Hence eqn (10) must be replaced by the Fadle-Papkovich eigenfunctions (Timoshenko and Goodier, 1970):

$$\psi^{(s)}(y) = A_1^{(s)} \cos\lambda y + A_2^{(s)} y \sin\lambda y + A_3^{(s)} \sin\lambda y + A_4^{(s)} y \cos\lambda y. \qquad (13)$$

Stress-strain relations can be written as:

$$\begin{aligned} u_{1,x}^{(s)}/h &= R_{11}^{(s)}\sigma_{11}^{(s)} + R_{12}^{(s)}\sigma_{22}^{(s)} \\ u_{2,y}^{(s)}/h &= R_{12}^{(s)}\sigma_{11}^{(s)} + R_{22}^{(s)}\sigma_{22}^{(s)} \\ (u_{1,y}^{(s)} + u_{2,x}^{(s)})/h &= R_{66}^{(s)}\sigma_{12}^{(s)}. \end{aligned} \qquad (14)$$

Moreover, a differential equation for the transverse displacement u_2 can be obtained from eqns (14a, 14c) in the form:

$$u_{2,xx}^{(s)}/h = R_{66}^{(s)}\sigma_{12,x}^{(s)} - R_{11}^{(s)}\sigma_{11,y}^{(s)} - R_{12}^{(s)}\sigma_{22,y}^{(s)}. \qquad (15)$$

Making use of eqns (9), the displacement components can be obtained by integrating eqns (14a) and (15), so obtaining

$$\begin{aligned} u_1^{(s)} &= -\lambda h e^{-\lambda x}\left(\frac{R_{11}^{(s)}}{\lambda^2}\frac{d^2\psi^{(s)}}{dy^2} + R_{12}^{(s)}\psi^{(s)}\right) \\ u_2^{(s)} &= -\lambda h e^{-\lambda x}\left(\frac{R_{11}^{(s)}}{\lambda^3}\frac{d^3\psi^{(s)}}{dy^3} + \frac{R_{66}^{(s)}+R_{12}^{(s)}}{\lambda}\frac{d\psi^{(s)}}{dy}\right) \end{aligned} \qquad (16)$$

to within a rigid body displacement. Traction-free boundary conditions on the external faces ($y = \pm 1$) and perfect bonding at the interfaces y_s ($s = 1, \ldots, S-1$) can be written as:

$$\sigma_{12}^{(1)}(x,-1)=\sigma_{22}^{(1)}(x,-1)=0,\ \sigma_{12}^{(S)}(x,1)=\sigma_{22}^{(S)}(x,1)=0, \tag{17}$$

$$\{\sigma_{12}^{(s)},\sigma_{22}^{(s)},u_1^{(s)},u_2^{(s)}\}=\{\sigma_{12}^{(s+1)},\ \sigma_{22}^{(s+1)},u_1^{(s+1)},u_2^{(s+1)}\}\quad \text{for } y=y_s. \tag{18}$$

Making use of eqns (9, 16), eqns (17, 18) give an homogeneous system of $4S$ equations for the $4S$ unknown coefficients $A_i^{(s)}$ associated with the eigenfunction $\psi^{(s)}(y)$ of eqns (10) or (12) (eigenvalue problem). Imposing the vanishing of the determinant of the coefficients yields the infinite set of eigenvalue-eigenvector couples $(\lambda; A_i^{(s)})$.

Finally, it can be verified that eqns (9) represent self-equilibrated stress distributions, so providing a convenient framework for investigating the decay of end effects. For instance, making use of eqns (17, 18) and integration by parts, the axial resultant becomes:

$$\int_{-1}^{1}\sigma_{11}\,\mathrm{d}y=e^{-\lambda x}\sum_{s=1}^{S}\left[\frac{\mathrm{d}\psi^{(s)}}{\mathrm{d}y}\right]_{y_{s-1}}^{y_s}=e^{-\lambda x}\left\{\frac{\mathrm{d}\psi^{(S)}}{\mathrm{d}y}\bigg|_{1}-\frac{\mathrm{d}\psi^{(1)}}{\mathrm{d}y}\bigg|_{-1}+\sum_{s=1}^{S-1}\left[\frac{\mathrm{d}\psi^{(s)}}{\mathrm{d}y}-\frac{\mathrm{d}\psi^{(s+1)}}{\mathrm{d}y}\right]_{y_s}\right\}=0. \tag{19}$$

The approach presented above for general multilayered strips will be used in the next sections to examine some interesting lamination cases, such as homogeneous and sandwich strips where simple characteristic eigenvalue equations can be obtained, and multilayered strips with periodic layout.

3. HOMOGENEOUS STRIP

For homogeneous (single layer) orthotropic strips only the free-side boundary conditions (17) must be imposed, and the eigenvalues λ are the nonzero roots of the transcendental equations (Choi and Horgan, 1977):

$$\frac{\sin c_1\lambda}{c_1}\pm\frac{\sin c_2\lambda}{c_2}=0. \tag{20}$$

Plus and minus signs give the eigenfunctions which are even and odd functions of the transverse coordinate y, corresponding to symmetric and antisymmetric deformations. In the isotropic case ($\bar{E}=2, \varepsilon=1/2$), the material constants (11) take the values $c_1=2$ and $c_2=0$, eqns (20) reduce to $\sin 2\lambda \pm 2\lambda=0$ and the classical Fadle-Papkovich eigenconditions are reobtained.

As pointed out in Horgan and Simmonds (1991) and Miller Horgan (1995a, b), although four elastic constants characterize an orthotropic strip, the eigenvalue problems (20) can be rewritten in terms of the combined elastic parameter ε of eqn (3a) only. In fact the eigenvalue λ can be replaced with $\lambda^*/\bar{E}^{1/2}$, so discharging $\bar{E}$ from the constants c_1, c_2 (see eqn 11). The first even eigenvalue λ^* is plotted in Crafter *et al.* (1993) and Wang *et al.* (1993) as a function of the combination of the elastic constants (different from ε). Plots of the first three even and odd eigenvalues λ^* vs ε may be found in Savoia and Tullini (1996).

When strongly orthotropic materials are concerned, i.e., for the ratio between shear modulus and axial Young's modulus approaching zero, the constant ε is close to zero and $c_1=c_2=\bar{E}^{1/2}$. Therefore, eqn (20) yields the eigenconditions $\sin\lambda^*=0$ and $\tan\lambda^*=\lambda^*$ for the even and odd eigenfunctions, respectively. The former equation has been already obtained by Choi and Horgan (1977), the latter by Horgan and Simmonds (1991) as the zero-order solution of their asymptotic analysis of an end-loaded transversely isotropic strip weak in shear and by Savoia and Tullini (1996) for the boundary problem of their beam theory. This solution is particularly interesting when eigenfunction expansions are used to solve boundary-value problems with prescribed stresses and/or displacements at the ends. In fact, these eigenfunctions constitute a classical Sturm-Liouville complete set of functions, satisfying simple orthogonality conditions, whereas more cumbersome techniques are required for the general case (e.g., see Horgan and Knowles, 1983; Horgan

1989; Kim and Steele, 1990; Lin and Wan, 1990). The numerical results obtained in previous investigations agree very well with exact and finite element solutions even in the neighbourhood of clamped cross-sections (Tullini and Savoia, 1995; Savoia and Tullini, 1996).

4. SANDWICH STRIP

A symmetric sandwich orthotropic strip is considered, where the external layers and the core of thicknesses $h^{(1)}$ and $h^{(2)}$ are made of two different materials denoted by superscripts (1) and (2). Setting:

$$\alpha = \frac{R_{11}^{(2)}}{R_{11}^{(1)}},\quad \beta = \frac{R_{66}^{(1)} - R_{66}^{(2)}}{R_{11}^{(1)}} \tag{21}$$

the eigenvalues λ of the sandwich strip are the nonzero roots of the transcendental equation:

$$\begin{aligned} 2[(c_1^{(2)}c_2^{(2)})^2\alpha^2 + \beta^2]d_1(\lambda)d_2(\lambda) + 4\alpha\beta d_1(\lambda)d_3(\lambda) \\ + \alpha[2d_3(\lambda)d_4(\lambda) + (c_1^{(1)^2} - c_2^{(1)^2})d_5(\lambda)d_6(\lambda) + (c_1^{(2)^2} - c_2^{(2)^2})d_7(\lambda)d_8(\lambda)] \\ + 2\beta d_2(\lambda)d_4(\lambda) + 2[c_1^{(1)^2} - c_2^{(1)^2} + (c_1^{(1)}c_2^{(1)})^2 d_1(\lambda)]d_2(\lambda) = 0 \end{aligned} \tag{22}$$

where:

$$\begin{aligned} d_1(\lambda) &= \sin^2(\lambda c_1^{(1)}h^{(1)}/H)/c_1^{(1)^2} - \sin^2(\lambda c_2^{(1)}h^{(1)}/H)/c_2^{(1)^2} \\ d_2(\lambda) &= \sin(\lambda c_1^{(2)}h^{(2)}/H)/c_1^{(2)} \pm \sin(\lambda c_2^{(2)}h^{(2)}/H)/c_2^{(2)} \\ d_3(\lambda) &= c_1^{(2)}\sin(\lambda c_1^{(2)}h^{(2)}/H) \pm c_2^{(2)}\sin(\lambda c_2^{(2)}h^{(2)}/H) \\ d_4(\lambda) &= \cos(\lambda c_1^{(1)}2h^{(1)}/H) - \cos(\lambda c_2^{(1)}2h^{(1)}/H) \\ d_5(\lambda) &= \cos(\lambda c_1^{(2)}h^{(2)}/H) \pm \cos(\lambda c_2^{(2)}h^{(2)}/H) \\ d_6(\lambda) &= \sin(\lambda c_1^{(1)}2h^{(1)}/H)/c_1^{(1)} + \sin(\lambda c_2^{(1)}2h^{(1)}/H)/c_2^{(1)} \\ d_7(\lambda) &= \cos(\lambda c_1^{(2)}h^{(2)}/H) \mp \cos(\lambda c_2^{(2)}h^{(2)}/H) \\ d_8(\lambda) &= \sin(\lambda c_1^{(1)}2h^{(1)}/H)/c_1^{(1)} - \sin(\lambda c_2^{(1)}2h^{(1)}/H)/c_2^{(1)}. \end{aligned} \tag{23}$$

In eqns (23) the upper sign corresponds to symmetric deformations of the middle layer, i.e., $A_2^{(2)} = A_4^{(2)} = 0$ in eqn (10), whereas the lower sign refers to antisymmetric deformations, $A_1^{(2)} = A_3^{(2)} = 0$. The governing eigenvalue problem (22) depends on one dimensionless height, e.g., $h^{(1)}/H$ as well as on the six material parameters $c_1^{(s)}$, $c_2^{(s)}$ $(s = 1, 2)$ and α, β defined in eqns (11,21) (cf. Ting, 1995). By replacing λ with $\lambda^*/(\bar{E}^{(1)})^{1/2}$ and discharging $\bar{E}^{(1)}$ from the constants $c_1^{(1)}$ and $c_2^{(1)}$, the eigenvalue equations can be rewritten in terms of five parameters only, i.e., α, β, $\bar{E}^{(2)}/\bar{E}^{(1)}$, $\varepsilon^{(1)}$, $\varepsilon^{(2)}$.

The elastic constants considered in the first numerical example, typical of a graphite-epoxy composite, are similar to those adopted in Miller and Horgan (1995):

$$\begin{aligned} &E_1^{(1)} = 127.5\,\text{Gpa},\ E_2^{(1)} = E_3^{(1)} = 11\,\text{GPa},\ G_{12}^{(1)} = 5.5\,\text{GPa},\ \nu_{12}^{(1)} = \nu_{13}^{(1)} = 0.35,\ \nu_{32}^{(1)} = 0.25 \\ &E_1^{(2)} = E_2^{(2)} = 11\,\text{GPa},\ E_3^{(2)} = 127.5\,\text{GPa},\ G_{12}^{(2)} = 4.4\,\text{GPa},\ \nu_{12}^{(2)} = 0.25,\ \nu_{31}^{(2)} = \nu_{32}^{(2)} = 0.35. \end{aligned} \tag{24}$$

Table 1. Eigenvalues for a sandwich strip [1-2-1] in plane strain.

	Symmetric		Antisymmetric	
Mode	Exact	FEM	Exact	FEM
1	1.718392	1.718450	1.340142	1.340151
2	1.886235	1.886210	2.666545	2.666792
	±0.395932*i*	±0.395912*i*		
3	2.659466	2.659775	4.348352	4.351796
4	4.103481	4.105499	4.378532	4.378044
			±0.891135*i*	±0.891208*i*
5	5.379101	5.387026	5.362104	5.370638
6	6.749806	6.753232	6.912725	6.939364
	±0.733040*i*	±0.740525*i*		
7	7.916500	7.963426	8.182118	8.194508
			±0.266809*i*	±0.280867*i*
8	8.060559	8.118900	9.060700	9.187669
9	9.311376	9.346139	10.767631	10.973767
	±0.708227*i*	±0.674485*i*		
10	10.719300	10.919044	11.261657	11.244595
			±1.644856*i*	±1.646681*i*

Table 1 contains the first 10 symmetric and antisymmetric eigenvalues λ for a three-layered strip in plane strain with external layer thickness $h^{(1)} = H/4$ and stacking sequence [1-2-1]. Displacement and stress components for the first even and odd eigenfunction are depicted in Figs 1, 2. Comparison is made with numerical results obtained through the F.E. technique proposed by Dong and Goetschel (1982), by adopting 40 elements over the strip height and quadratic interpolation polynomials (162 degrees of freedom). Table 1 shows that FEM technique gives very accurate results, the error being less than 2% for all the eigenvalues computed. Observe that the eigenvalue of smallest real part corresponds to the antisymmetric deformation.

For a sandwich strip with isotropic layers ($\bar{E}^{(s)} = 2$, $\varepsilon^{(s)} = 1/2$), the eigenvalue equation can be found from eqns (22, 23) for $c_1^{(s)} \to 2$ and $c_2^{(s)} \to 0$, so obtaining:

$$\beta^2 d_1(\lambda) d_2(\lambda) + 8\alpha\beta d_1(\lambda)\sin(\lambda 2h^{(2)}/H) + 16\alpha[\sin 2\lambda \pm 2\lambda - d_2(\lambda)] - 8\beta d_2(\lambda)\sin^2(\lambda 2h^{(1)}/H) + 16 d_2(\lambda) = 0 \quad (25)$$

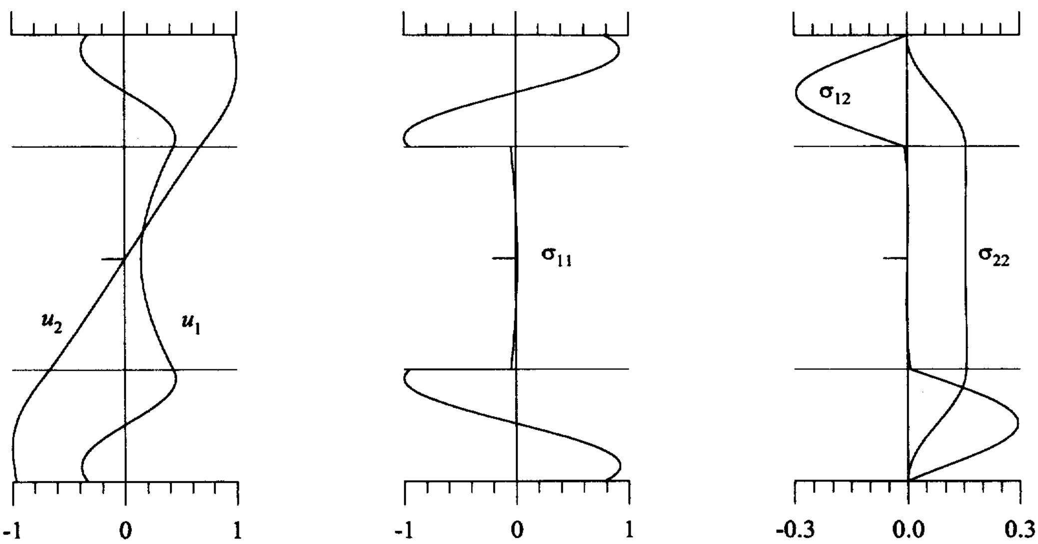

Fig. 1. First even eigenfunction of a sandwich strip [1-2-1]. (a) Axial and transverse displacement; (b, c) Stress components. Stresses are reported to the accuracy of $\lambda^2 e^{-\lambda x}$ and scales by 1/93.36.

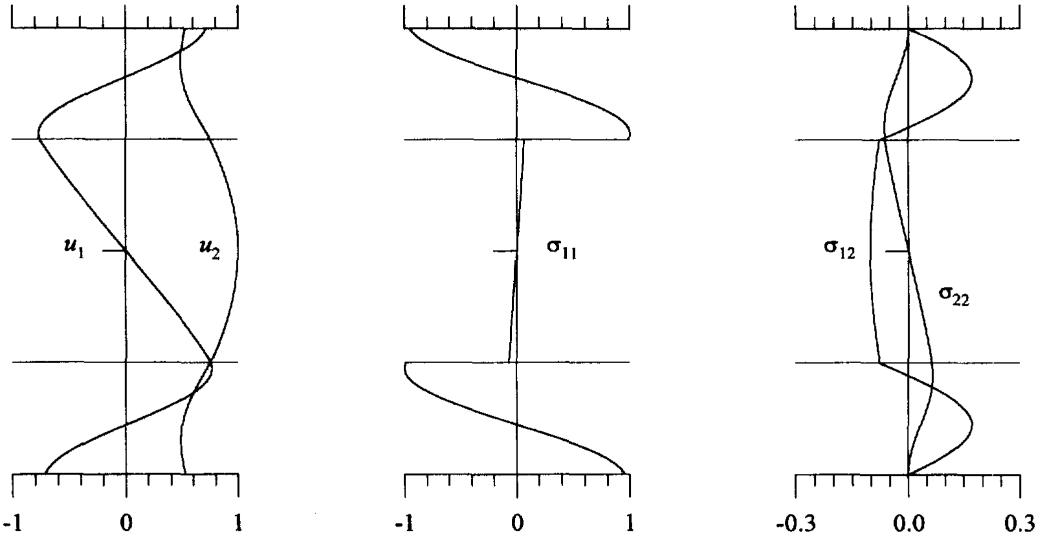

Fig. 2. First odd eigenfunction of a sandwich strip [1-2-1]. (a) Axial and transverse displacement; (b, c) Stress components. Stresses are reported to the accuracy of $\lambda^2 e^{-\lambda x}$ and scales by 1/129.04.

where:

$$d_1(\lambda) = \sin^2(\lambda 2h^{(1)}/H) - (\lambda 2h^{(1)}/H)^2, \quad d_2(\lambda) = \sin(\lambda 2h^{(2)}/H) \pm (\lambda 2h^{(2)}/H). \tag{26}$$

It is easy to verify that eqn (25) is equivalent to eqn (3.5) reported in Choi and Horgan (1978).

Of course, eigenvalue eqn (25) can be written with reference to material parameters different from those reported in eqns (21). For instance, Wijeyewickrema (1995) and Wijeyewickrema *et al.* (1996) showed that the two material constants proposed by Dundurs (1969) offer considerable simplification in the analysis of end effects for isotropic sandwich strips. In the isotropic case, the relation between Dundurs' constants α_D, β_D and α and β of eqns (21) are:

$$\alpha_D = \frac{1-\alpha}{1+\alpha}, \quad \beta_D = \frac{1-\alpha-\beta/4}{1+\alpha}. \tag{27}$$

5. STRONGLY ORTHOTROPIC SANDWICH STRIP

The eigenvalue problem (22) can be simplified if the layers are made of strongly orthotropic materials. In this case, the ratios between shear and axial Young's moduli approach zero together with the constants $\varepsilon^{(s)}$, so that $c_1^{(s)} = c_2^{(s)} = (\bar{E}^{(2)})^{1/2} = c^{(s)}$. By using an appropriate Taylor expansion of eqn (22) with respect to $\varepsilon^{(s)}$ and introducing the combined strip parameters $\Lambda^{(1)} = \lambda c^{(1)} 2h^{(1)}/H$, $\Lambda^{(2)} = \lambda c^{(2)} h^2/H$, the eigenvalue problem (22) reduced to:

$$S)\ c^{(1)}/R^{(1)}_{66} \cos\Lambda^{(1)} \sin\Lambda^{(2)} + c^{(2)}/R^{(2)}_{66} \sin\Lambda^{(1)} \cos\Lambda^{(2)} = 0 \tag{28}$$

$$\begin{aligned} A)\ & c^{(1)} \cos\Lambda^{(1)} \sin\Lambda^{(2)} + c^{(2)} R^{(2)}_{66}/R^{(1)}_{66} \sin\Lambda^{(1)} \cos\Lambda^{(2)} \\ & + 2c^{(1)} R^{(2)}_{66}/R^{(1)}_{66} (1 - R^{(2)}_{66}/R^{(1)}_{66})(1 - \cos\Lambda^{(1)}) \sin\Lambda^{(2)} - (h^{(2)}/H + R^{(2)}_{66}/R^{(1)}_{66} 2h^{(1)}/H) \\ & \times \lambda c^{(1)} (c^{(2)} \cos\Lambda^{(1)} \cos\Lambda^{(2)} - c^{(1)} R^{(2)}_{66}/R^{(1)}_{66} \sin\Lambda^{(1)} \sin\Lambda^{(2)}) = 0 \end{aligned} \tag{29}$$

for symmetric and antisymmetric deformations, respectively. It is easy to show that eigenconditions (28, 29) can be written as a function of two material parameters only. For instance, the following new eigenvalue and material parameters can be employed:

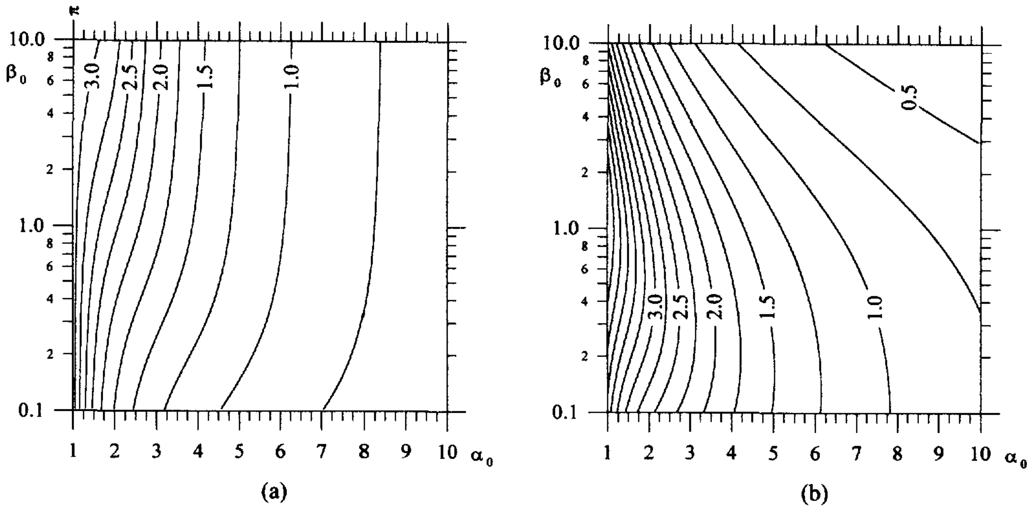

Fig. 3. Contour lines for (a) even and (b) odd eigenvalues λ^* (see eqn 30a) vs material parameters α_0, β_0 reported in eqns (30b, c) for a strongly orthotropic sandwich strip with layer thicknesses $[H/4, H/2, H/4]$.

$$\lambda^* = \lambda c^{(2)}, \quad \alpha_0 = c^{(1)}/c^{(2)}, \quad \beta_0 = R_{66}^{(2)}/R_{66}^{(1)}. \tag{30}$$

In Figs 3a, b the contour lines of even and odd (real) eigenvalues λ^* are plotted as a function of α_0, β_0 for a strongly orthotropic sandwich strip in plane strain with external layer thickness $h^{(1)} = H/4$.

It is interesting to assess the range of validity of approximate eigenconditions (28, 29), that is the validity of the 'strong orthotropy' hypothesis. To this purpose, a sandwich strip in plane strain with stacking sequence [1-2-1] is considered. The elastic constants adopted, typical of a high-strength graphite-epoxy composite with cross-ply lamination scheme [0/90/0] are (Pagano, 1969):

$$(1)\ E_1^{(1)} = 175\,\text{GPa},\ E_2^{(1)} = E_3^{(1)} = 7\,\text{GPa},\ G_{12}^{(1)} = 3.5\,\text{GPa},\ \nu_{12}^{(1)} = \nu_{13}^{(1)} = \nu_{32}^{(1)} = 0.25$$

$$(2)\ E_1^{(2)} = E_2^{(2)} = 7\,\text{GPa},\ E_3^{(2)} = 175\,\text{GPa},\ G_{12}^{(2)} = 1.4\,\text{GPa},\ \nu_{12}^{(2)} = \nu_{31}^{(2)} = \nu_{32}^{(2)} = 0.25. \tag{31}$$

The parameters $\bar{E}$ and ε of eqns (7) corresponding to materials (1) and (2) are summarized in Table 2, together with the first eigenvalue of homogeneous strips made of the two different materials. These eigenvalues are compared with those obtained from the 'strong orthotropy' hypothesis ($\varepsilon = 0$). As is to be expected, material (1) can be considered strongly orthotropic, the error being less 1.5%, whereas for the inner layer the error is 9.3% for the first symmetric eigenfunction. Finally, for the sandwich strip, Fig. 4 shows the first even and odd (real) exact eigenvalues and those obtained with the 'strong orthotropy' hypothesis, as a function of the external layer thickness $h^{(1)}/H$. The maximum error is 14.6% and occurs at $h^{(1)}/H = 0.195$ for the symmetric eigenfunction. The error is much smaller for the antisymmetric deformation (6.5% at $h^{(1)}/H = 0.1025$). From Fig. 4, it can be seen that the first eigenvalue for symmetric deformation is smaller than that for antisymmetric deformation for $0.165 < h^{(1)}/H < 0.435$. Thus, in this range for $h^{(1)}/H$, the decay length for the symmetric deformation is larger than that for antisymmetric deformation.

Table 2. First eigenvalue (real) λ_1 for homogeneous orthotropic strips in plane strain

Material	$\bar{E}$	ε	Symmetric Exact	Symmetric $\varepsilon = 0$	Diff.%	Antisymmetric Exact	Antisymmetric $\varepsilon = 0$	Diff.%
(1)	49.49875	0.09793	0.453292	0.446532	−1.49	0.643923	0.638673	−0.81
(2)	4.50627	0.22191	1.632086	1.479930	−9.32	2.218750	2.116740	−4.50

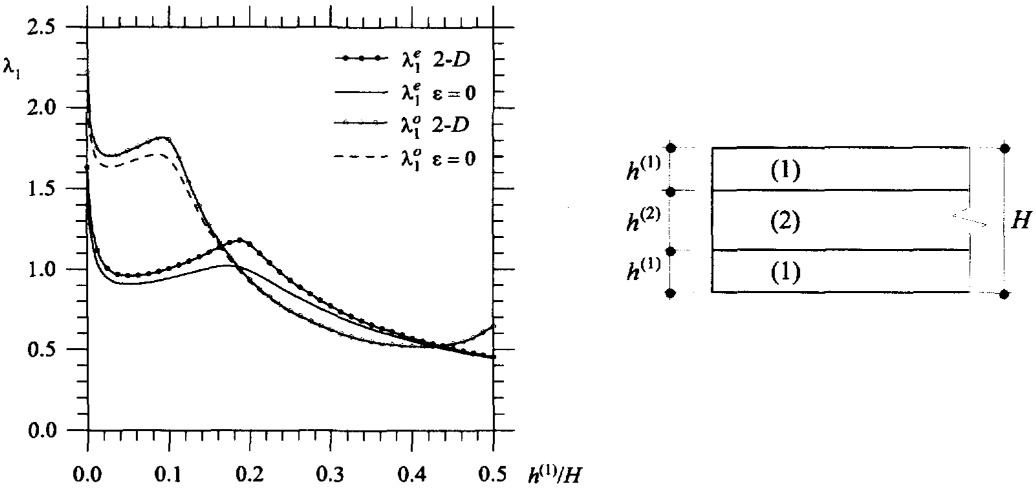

Fig. 4. First even and odd exact eigenvalues of a sandwich strip vs external layer thickness $h^{(1)}/H$, compared with those obtained from the 'strong orthotropy' assumption.

6. LAMINATED STRIP WITH PERIODIC LAYOUT

6.1. *Equivalent elastic coefficients via homogenization theory*

Homogenization theory can be used in the study of partial differential equations with rapidly varying coefficients. For periodic composite materials, the homogenization method gives the effective elastic moduli of an homogeneous material whose overall response is 'close' to that of the heterogeneous periodic material when the size of the elementary cell of periodicity, depending on a small parameter ε, tends to zero. It allows also for the a-posteriori computation of local stress variation at the layer level. For more information about homogenization theory the reader is referred to Bensoussan *et al.* (1978), Sanchez-Palencia (1980) and Oleinik *et al.* (1992).

The effective elastic coefficients of a layered strip can be obtained by homogenization of the fourth-order operator (2) with boundary conditions (17, 18). These equations can be rewritten in the compact form ($\alpha, \beta, \ldots = 1, 2$):

$$(\alpha^{\varepsilon}_{\alpha\beta\gamma\delta} F^{\varepsilon}_{,\gamma\delta})_{,\alpha\beta} = 0 \qquad x_1 > 0,\ -h < x_2 < h \tag{32}$$

$$F^{\varepsilon} = F^{\varepsilon}_{,2} = 0 \qquad x_2 = \pm h \tag{33}$$

$$F^{\varepsilon} = g_0,\ F^{\varepsilon}_{,1} = g_1 \qquad x_1 = 0 \tag{34}$$

where the summation convention is used and partial differentiation must be considered in the weak sense (virtual force principle), because the coefficients $a^{\varepsilon}_{\alpha\beta\gamma\delta}$ are piecewise constants and periodic, with period of the order of ε, but traction and displacement continuity (18) must hold at the interfaces (essential and natural boundary conditions, respectively). For orthotropic layers, the non vanishing coefficients $\alpha^{\varepsilon}_{\alpha\beta\gamma\delta}$ in eqn (32) are:

$$a^{\varepsilon}_{1111} = R^{\varepsilon}_{22},\ a^{\varepsilon}_{2222} = R^{\varepsilon}_{11},\quad a^{\varepsilon}_{1122} = R^{\varepsilon}_{12},\ a^{\varepsilon}_{1212} = R^{\varepsilon}_{66}/4. \tag{35}$$

Further, g_0 and g_1 are prescribed functions at the strip end $x_1 = 0$.

The homogenization process is first performed with reference to the more general case of a composite domain with periodicity in both directions x_1 and x_2. Consider a bounded Lipschitz domain Ω of the space $\mathbb{R}^2$, whose points are denoted by $\mathbf{x} = (x_1, x_2)$ (global coordinates). An auxiliar space $\mathbb{R}^2$ is introduced, with points $\mathbf{y} = (y_1, y_2)$ (local coordinates), and the single rectangular cell of periodicity is denoted by $Y =]0, Y_1[\times]0, Y_2[$ (the representative volume element, RVE). Let $a_{\alpha\beta\gamma\delta}(\mathbf{y})$ be bounded and Y-periodic functions satisfying the symmetry conditions $a_{\alpha\beta\gamma\delta}(\mathbf{y}) = a_{\beta\alpha\gamma\delta}(\mathbf{y}) = a_{\gamma\delta\alpha\beta}(\mathbf{y})$ and the inequalities:

$$mS_{\alpha\beta}S_{\alpha\beta} \leqslant a_{\alpha\beta\gamma\delta}(\mathbf{y})S_{\alpha\beta}S_{\gamma\delta} \leqslant MS_{\alpha\beta}S_{\alpha\beta} \quad \forall \mathbf{S} \in \mathrm{sym}(\mathbb{R}^2 \otimes \mathbb{R}^2) \tag{36}$$

where m, M are positive constants. Hence, introducing the coordinate transformation $\mathbf{x} := \varepsilon\mathbf{y}$, where $\varepsilon > 0$ is a characteristic (small) parameter denoting the dimension of the cell εY, the coefficients $a^{\varepsilon}_{\alpha\beta\gamma\delta}$ are εY-periodic over Ω and defined as:

$$a^{\varepsilon}_{\alpha\beta\gamma\delta}(\mathbf{x}) := a_{\alpha\beta\gamma\delta}\left(\frac{\mathbf{x}}{\varepsilon}\right). \tag{37}$$

For a periodic medium, the following problem governed by a fourth-order partial differential operator with Dirichlet boundary conditions is considered:

$$\begin{cases} \text{Given } a^{\varepsilon}_{\alpha\beta\gamma\delta} \text{ bounded and } \varepsilon Y\text{-periodic}, f \in L^2(\Omega),\ g_0 \in H^{3/2}(\Omega),\ g_1 \in H^{1/2}(\Omega) \\ \text{find } F^{\varepsilon} \in H^2(\Omega) \text{ such that:} \\ (a^{\varepsilon}_{\alpha\beta\gamma\delta}(\mathbf{x})F^{\varepsilon}_{,\gamma\delta})_{,\alpha\beta} = f(\mathbf{x}) \ \text{ on } \Omega \\ F^{\varepsilon} = g_0,\ F^{\varepsilon}_{,n} = g_1 \ \text{ on } \partial\Omega \end{cases} \tag{38}$$

where n is the outward normal to $\partial\Omega$, L^2, H^2 are the Sobolev spaces, $H^{1/2}$ and $H^{3/2}$ are trace spaces. Trace theorem assures the existence of a function $F_* \in H^2(\Omega)$ such that $F_* = g_0$, $F_{*,n} = g_1$ on $\partial\Omega$ (see e.g., Reddy, 1986).

Hence, problem (38) admits the following variational formulation for $\varepsilon > 0$:

$$\begin{cases} \text{Find } F^{\varepsilon}_d := F^{\varepsilon} - F_* \in H^2_0(\Omega) \ \text{ such that } \ \forall \tilde{F} \in H^2_0(\Omega) \\ \displaystyle\int_{\Omega} a^{\varepsilon}_{\alpha\beta\gamma\delta} F^{\varepsilon}_{d,\gamma\delta} \tilde{F}_{,\alpha\beta}\,\mathrm{d}\Omega = \int_{\Omega} f\tilde{F}\,\mathrm{d}\Omega - \int_{\Omega} a^{\varepsilon}_{\alpha\beta\gamma\delta} F_{*,\gamma\delta}\tilde{F}_{,\alpha\beta}\,\mathrm{d}\Omega \end{cases} \tag{39}$$

It can be shown that, as $\varepsilon \to 0$, the solution F^{ε}_d of (39) converges weakly in $H^2_0(\Omega)$ to F^0_d, unique solution of the homogenized problem:

$$\begin{cases} \text{Find } F^0_d := F^0 - F_* \in H^2_0(\Omega) \ \text{ such that } \ \forall \tilde{F} \in H^2_0(\Omega) \\ \displaystyle\hat{a}_{\alpha\beta\gamma\delta}\int_{\Omega} F^0_{d,\gamma\delta} \tilde{F}_{,\alpha\beta}\,\mathrm{d}\Omega = \int_{\Omega} f\tilde{F}\,\mathrm{d}\Omega - \hat{a}_{\alpha\beta\gamma\delta}\int_{\Omega} F_{*,\gamma\delta}\tilde{F}_{,\alpha\beta}\,\mathrm{d}\Omega \end{cases} \tag{40}$$

where the homogenized coefficients $\hat{a}_{\alpha\beta\gamma\delta}$ are given by:

$$\hat{a}_{\alpha\beta\gamma\delta} = \frac{1}{|Y|}\int_Y (a_{\alpha\beta\gamma\delta} - a_{\mu\nu\gamma\delta} w_{\alpha\beta|\mu\nu})\,\mathrm{d}Y \tag{41}$$

and $w_{\alpha\beta}(\mathbf{y})$, the homogenization functions, are the (unique) solution of the following variational problem (the cell problem):

$$\begin{cases} \text{Find } w_{\alpha\beta} \in H^2_{\mathrm{per}}(Y) \text{ such that } \forall v \in H^2_{\mathrm{per}}(Y) \\ \displaystyle\int_Y (a_{\alpha\beta\gamma\delta} - a_{\mu\nu\gamma\delta} w_{\alpha\beta|\mu\nu}) v_{|\gamma\delta}\,\mathrm{d}Y = 0 \end{cases} \tag{42}$$

where $(\bullet)_{|\alpha}$ denotes partial derivative with respect to y_α and

$$H^2_{\text{per}}(Y) := \{v \in H^2(Y) | v \text{ and } v_{|\alpha} \text{ } Y\text{-periodic and } \int_Y v \, dY = 0\}. \tag{43}$$

The proof of this theorem is analogous to that of homogenization of the second-order elliptic equation and of the elasticity problem (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980). Similar theorem statement is given in Duvaut (1976) for isotropic plates and in Caillerie (1984, 1987) for anisotropic plates with extension-bending coupling, two problems governed by a differential equation similar to eqn (32).

The theorem suggests the possibility of assuming, for small values of ε, the following two-scale asymptotic expansion for the solution F^ε of the problem (39) (Kohn and Vogelius, 1984; Lewinski, 1991):

$$F^\varepsilon(\mathbf{x}) = F^0(\mathbf{x}) - \varepsilon^2 F^0_{,\alpha\beta}(\mathbf{x}) w_{\alpha\beta}(\mathbf{x}/\varepsilon) + O(\varepsilon^4). \tag{44}$$

This statement can be easily verified. Performing the second derivative of F^ε, multiplying by $a^\varepsilon_{\alpha\beta\gamma\delta}$ and after some algebraic manipulations yield:

$$a^\varepsilon_{\alpha\beta\gamma\delta} F^\varepsilon_{d,\gamma\delta} = F^0_{d,\alpha\beta}(a_{\alpha\beta\gamma\delta} - a_{\mu\nu\gamma\delta} w_{\alpha\beta|\mu\nu}) - F_{*,\alpha\beta}(a_{\mu\nu\gamma\delta} w_{\alpha\beta|\mu\nu}) + O(\varepsilon). \tag{45}$$

Hence, substituting eqn (45) in (39), neglecting terms of higher power of ε, taking the arithmetic mean over the period Y and assuming the test function $\tilde{F}$ independent of ε, the homogenized problem (40) with coefficients (41) is reobtained.

In addition, if the test function is taken in the form $\tilde{F} = \theta(\mathbf{x}) v(\mathbf{x}/\varepsilon)$ with $\theta \in C^\infty_0(\Omega)$ and $v \in H^2_{\text{per}}(Y)$, performing the arithmetic mean of eqn (39) over Y yields:

$$\int_\Omega F^0_{,\alpha\beta} \left[\int_Y (a_{\alpha\beta\gamma\delta} - a_{\mu\nu\gamma\delta} w_{\alpha\beta|\mu\nu}) v_{|\gamma\delta} \, dY \right] \theta \, d\Omega + O(\varepsilon^3) = \varepsilon^2 \int_\Omega f\theta v \, d\Omega. \tag{46}$$

Because θ is arbitrary implies that the expression in the square brackets of eqn (46) is identically zero to the accuracy of higher order terms of ε, so obtaining the governing equation of the cell problem (42).

Moreover, starting from eqn (36) it can be proved that the homogenized coefficients satisfy the symmetry conditions $\hat{a}_{\alpha\beta\gamma\delta} = \hat{a}_{\beta\alpha\gamma\delta} = \hat{a}_{\gamma\delta\alpha\beta}$ and that two positive constants $\hat{m}$, $\hat{M}$ exist, such that a condition for them analogous to eqn (36) holds. Hence, Lax-Milgram theorem proves the existence and uniqueness of the solution of problem (40) (see, e.g., Reddy, 1986).

6.2. *Periodically laminated composite strip*

For a composite medium periodic in one direction only, the homogenization process can be performed analytically. If $a_{\alpha\beta\gamma\delta}$ are independent of y_1, the RVE reduces to $Y =]0, Y_2[$ and the cell problem (42) gives the following differential equation:

$$(a_{\alpha\beta22} - a_{2222} w_{\alpha\beta|22})_{|22} = 0 \quad \text{on }]0, Y_2[\tag{47}$$

in the weak sense, with Y_2-periodic boundary conditions for the function enclosed in parentheses and for its first derivative with respect to y_2. Double-integrating eqn (47) and imposing the periodic boundary conditions, the following equation is obtained:

$$w_{\alpha\beta|22} = (a_{\alpha\beta22} - C_{\alpha\beta})/a_{2222} \tag{48}$$

where $C_{\alpha\beta}$ can be obtained from the condition $\langle w_{\alpha\beta|22} \rangle = [w_{\alpha\beta|2}]_0^{Y_2} = 0$ in the form:

$$C_{\alpha\beta} = \langle a_{\alpha\beta 22} a_{2222}^{-1} \rangle \langle a_{2222}^{-1} \rangle^{-1} \tag{49}$$

where $\langle \cdot \rangle$ denotes the arithmetic mean over Y_2:

$$\langle f \rangle = \frac{1}{|Y_2|} \int_0^{Y_2} f \mathrm{d}y_2. \tag{50}$$

Finally, the homogenized coefficients can be explicitly obtained from eqn (41, 48) as:

$$\hat{a}_{\alpha\beta\gamma\delta} = \langle a_{\alpha\beta\gamma\delta} - a_{22\gamma\delta} w_{\alpha\beta|22} \rangle = \langle a_{\alpha\beta\gamma\delta} - a_{\alpha\beta 22} a_{2222}^{-1} a_{22\gamma\delta} \rangle + C_{\alpha\beta} \langle a_{22\gamma\delta} a_{2222}^{-1} \rangle. \tag{51}$$

For the composite strip with orthotropic layers, coefficients $a_{\alpha\beta\gamma\delta}$ are related to the elastic moduli R_{ij} through eqn (35). Hence, eqn (51a) gives the following expressions for the non vanishing effective elastic moduli:

$$\hat{R}_{11} = \hat{a}_{2222} = \langle R_{11}^{-1} \rangle^{-1}, \ \hat{R}_{22} = \hat{a}_{1111} = \langle R_{22} - R_{12}^2 R_{11}^{-1} \rangle + \langle R_{12} R_{11}^{-1} \rangle^2 \langle R_{11}^{-1} \rangle^{-1}$$
$$\hat{R}_{12} = \hat{a}_{1122} = \langle R_{12} R_{11}^{-1} \rangle \langle R_{11}^{-1} \rangle^{-1}, \ \hat{R}_{66} = 4\hat{a}_{1212} = \langle R_{66} \rangle. \tag{52}$$

Analogously, derivatives of nonvanishing homogenization functions (48) are given by:

$$w_{11|22}(y_2) = [R_{12}(y_2) - \hat{R}_{12}]/R_{11}(y_2)$$
$$w_{22|22}(y_2) = 1 - \hat{R}_{11}/R_{11}(y_2). \tag{53}$$

Finally, local stress distributions are obtained from eqns (1), (44) and (53):

$$\sigma_{11}^{\varepsilon} = F_{,22}^{\varepsilon} = F_{,22}^{0} \hat{R}_{11}/R_{11}(y_2) - F_{,11}^{0}[R_{12}(y_2) - \hat{R}_{12}]/R_{11}(y_2) + O(\varepsilon)$$
$$\sigma_{12}^{\varepsilon} = -F_{,12}^{\varepsilon} = -F_{,12}^{0} + O(\varepsilon), \ \sigma_{22}^{\varepsilon} = F_{,11}^{\varepsilon} = F_{,11}^{0} + O(\varepsilon). \tag{54}$$

These equations show that transverse shear and normal stresses depend on the stress function F^0 of the homogenized material only and, correspondingly, no discontinuities are present at the layer interfaces. On the contrary, the local values of elastic moduli $R_{11}(y_2)$ and $R_{12}(y_2)$ are present in the computation of axial normal stresses, which are discontinuous at the layer interfaces, as is required by the elasticity problem.

6.3. *Comparison between exact and homogenized solution*

The exact eigenvalues of a multilayered strip with periodic layout are compared with those obtained by the homogenization method.

The first example refers to a multilayered strip in plane strain, whose RVE consists of a sandwich strip [1-2-1] with elastic constants reported in eqn (24) and layer thicknesses [1/4, 1/2, 1/4] of the height of the elementary cell. The strip is made of n_c elementary cells;

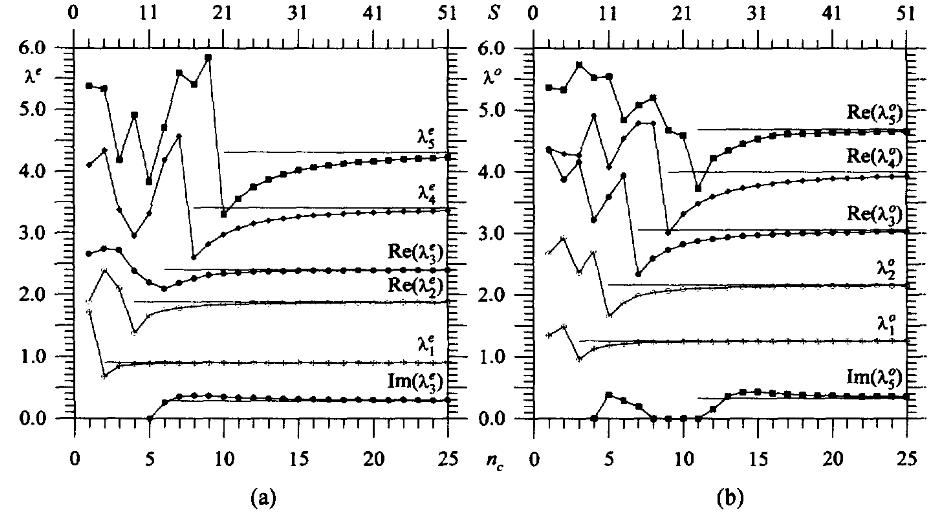

Fig. 5. (a) Even and (b) odd eigenvalues for strips in plane strain vs number n_c of elementary cells of periodicity, which are sandwich strips [1-2-1] with layer thicknesses [1/4, 1/2, 1/4] of the height of the elementary cell.

the total number of layers is $S = 2n_c+1$ and the stacking layout is symmetric. Figure 5 shows the real and imaginary parts of the first five even (λ^e) and odd (λ^o) eigenvalues as a function of the number of cells. The eigenvalues λ_2^e, λ_3^o and λ_4^o are complex for $n_c = 1, 2$, $n_c = 2$ and $n_c = 1, 3, 4$, respectively; their imaginary parts are not plotted for the sake of clearness. Making use of eqns (52), the homogenized material parameters are $\hat{\bar{E}} = 13.610$ and $\hat{\varepsilon} = 0.1797$, whereas for the individual layers $\bar{E}^{(1)} = 22.545$, $\varepsilon^{(1)} = 0.1470$ and $\bar{E}^{(2)} = 2$, $\varepsilon^{(2)} = 1/2$. The eigenvalues $\hat{\lambda}$ of the homogenized strip obtained from eqn (20) are reported in Fig. 5 by straight lines. It is worth noting that both real and imaginary parts of the exact eigenvalues approach asymptotically those of the homogenized strip as the number of elementary cells n_c increases. Moreover, the convergence turns out to be monotone if the number of cells is not too small, i.e., for $n_c \geqslant 2j$ ($n_c \geqslant 2j+1$) for the jth even (odd) eigenvalue.

A classical problem in homogenization theory is the computation of upper bounds for the error related with the homogenization process. For instance, upper bounds for eigenvalues of vibration problems have been obtained in Sanchez-Palencia (1980) and Oleinik *et al.* (1992). For the problem at hand, the relative error $\Delta_j = |\lambda_j - \hat{\lambda}_j|/|\hat{\lambda}_j|$ of the homogenized eigenvalues $\hat{\lambda}_j$ with respect to the first two even and odd exact eigenvalues is depicted in Figs 6a, b. In this case, the analytical estimate of the error is not available. However,

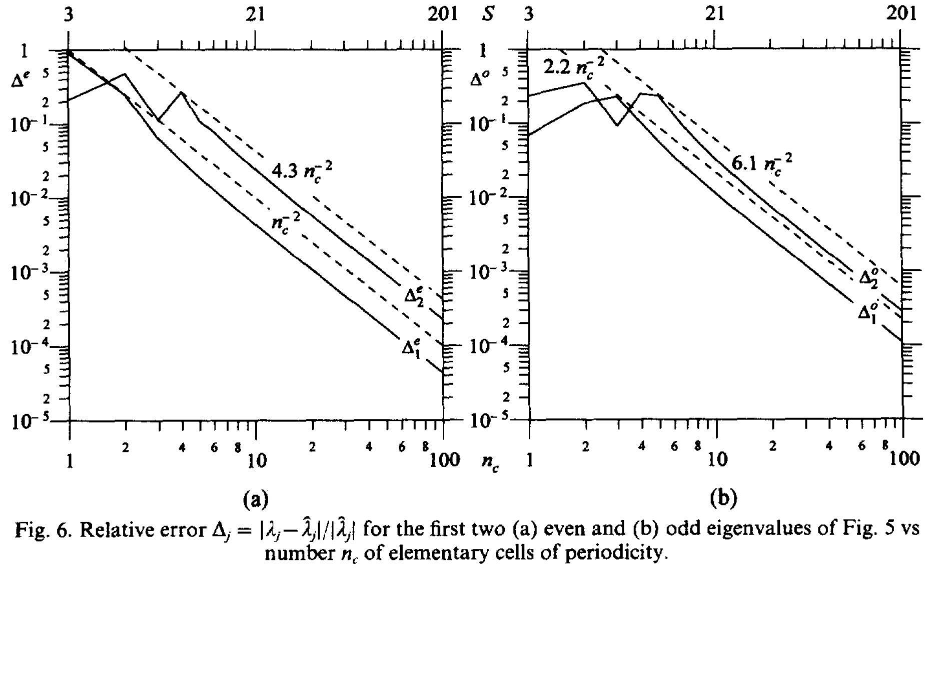

Fig. 6. Relative error $\Delta_j = |\lambda_j - \hat{\lambda}_j|/|\hat{\lambda}_j|$ for the first two (a) even and (b) odd eigenvalues of Fig. 5 vs number n_c of elementary cells of periodicity.

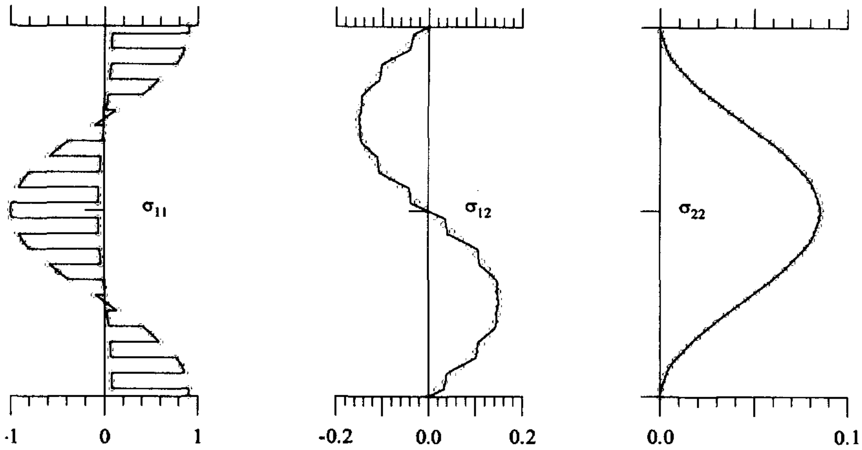

Fig. 7. Stress distributions corresponding to the first even eigenfunction for a strip composed by $n_c = 12$ cells of periodicity [1-2-1] with layer thicknesses [1/4, 1/2, 1/4] (25 layers) :—Exact elasticity solution ; ∘∘∘ Homogenized solution.

numerical computations suggest the validity of the following upper bound for the relative error Δ_j:

$$\Delta_j = \frac{|\lambda_j - \hat{\lambda}_j|}{|\hat{\lambda}_j|} \leqslant C_j n_c^{-2} \tag{55}$$

where C_j is a constant independent of the number of cells n_c whose value increases with the number j of the eigenvalue considered. This estimate is shown in Fig. 6 by a dashed line. Even though, at the moment, this result cannot be given a general validity, it appeared to be independent of the relative thickness and material properties of the layers. For instance, analogous results have been obtained for periodically laminated strips with elastic constants (31).

The stress distributions corresponding to the first even eigenfunction for a periodically laminated strip with 25 layers ($n_c = 12$) and computed through eqns (54) are reported in Fig. 7. It is confirmed that homogenization theory gives discontinuous results at the interlaminae for the axial normal stress σ_{11} and continuous distributions of transverse shear and normal stresses σ_{12}, σ_{22}. The results agree very well with those obtained through the elasticity solution.

The second example refers to FRP-wood multilayered strips in plane stress. The elastic moduli adopted for S-glass fibre-reinforced epoxy lamina (material 1) and wood (2) are:

$$(1)\ E_1^{(1)} = 50\,\text{GPa},\ E_2^{(1)} = 12\,\text{GPa},\ G_{12}^{(1)} = 5.5\,\text{GPa},\ \nu_{12}^{(1)} = \nu_{13}^{(1)} = \nu_{32}^{(1)} = 0.25$$

$$(2)\ E_1^{(2)} = 10\,\text{GPa},\ E_2^{(2)} = 0.8\,\text{GPa},\ G_{12}^{(2)} = 0.5\,\text{GPa},\ \nu_{12}^{(2)} = 0.35,\ \nu_{13}^{(2)} = 0.4,\ \nu_{32}^{(2)} = 0.3. \tag{56}$$

Figure 8a shows the first five eigenvalues λ vs the number n_c of elementary cells of periodicity, which are constituted by bimaterial strips [1-2] with nondimensional layer thicknesses [1/10, 9/10] of the height of the elementary cell. The total number of layers is $S = 2n_c$ and the stacking layout is asymmetric. Making use of eqns (52), the effective material constants (3) are $\hat{E} = 24.775$ and $\hat{\varepsilon} = 0.1606$, whereas for the individual layers $\bar{E}^{(1)} = 8.591$, $\varepsilon^{(1)} = 0.2376$ and $\bar{E}^{(2)} = 19.300$, $\varepsilon^{(2)} = 0.1832$. The eigenvalues of the homogenized material are obtained from eqns (20) and are reported in Fig. 8a by straight line. For asymmetric layouts, symmetric and antisymmetric deformation modes cannot be separated, but they converge to even and odd eigenvalues of the homogenized material. The relative error Δ_j (see eqn 55) for the first two eigenvalues is reported in Fig. 8b. As in the previous example, both real and imaginary parts of exact eigenvalues approach asymptotically those of the

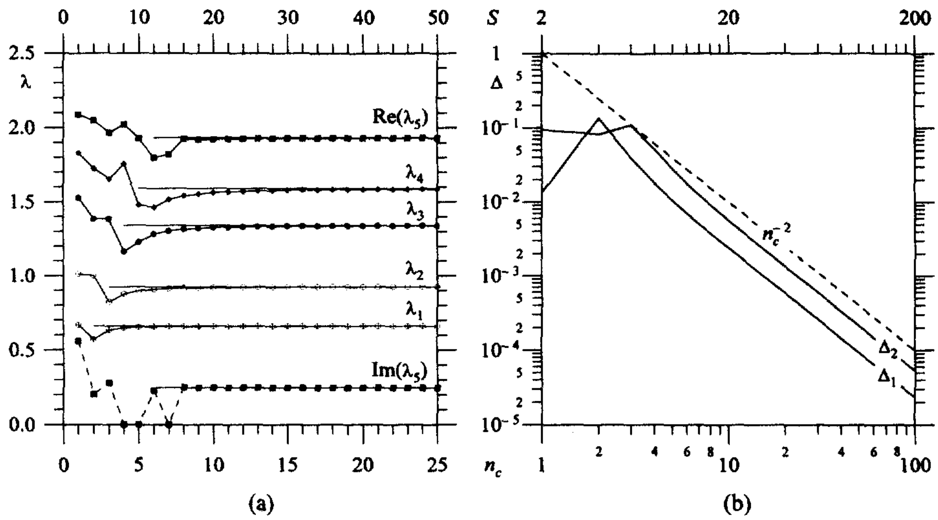

Fig. 8. (a) Eigenvalues λ_j and (b) relative errors Δ_j for strips in plane stress vs number n_c of elementary cells of periodicity, which are bimaterial strips [1-2] with layer thicknesses [1/10, 9/10] of the height of the elementary cell.

homogenized strip as the number n_c of elementary cells tends to infinity, and the order of the relative error is $O(n_c^{-2})$.

The numerical results assure that the effective reduced elastic coefficients $\hat{R}_{ij}$ defined in eqn (52) together with eigenconditions (20) give excellent estimates of eigenvalues of multilayered periodic strips. For the first eigenvalues, the approximation is good also when the number of layers S is relatively low. For instance, with reference to the example of Fig. 6, the error Δ_1 is less than 5% for $S \geqslant 9$ ($S \geqslant 11$) for the first even (odd) eigenvalue whereas, for that of Fig. 8b, the error is $\Delta_1 \leqslant 5\%$ for $S \geqslant 6$.

7. SOME REMARKS ABOUT THE HOMOGENIZATION PROCESS

The results presented are correct for multilayered strips under plane strain, whereas plane stress conditions must be treated much more carefully. In fact, the presence of free-edge effects at the lateral sides is very important for laminated strips, as can be inferred from the large amount of papers devoted to this topic. Hence, it is probably quite unrealistic to perform the homogenization process by considering that plane stress hypothesis holds a-priori for each individual layer.

A more correct approach should require the deduction of coefficients $\hat{R}_{ij}$ from a three-dimensional homogenized constitutive law. In this case, the effective elastic coefficients $\hat{C}_{ijhk}$ for an orthotropic medium periodically-laminated in the x_2-direction have been computed analytically (Oleinik *et al.* 1992, corollary 7.14, cap. II):

$$\hat{C}_{1111} = k_{11} + k_{13}^2 k_{22}^{-1},\quad \hat{C}_{2222} = k_{22}^{-1},\quad \hat{C}_{3333} = k_{33} + k_{23}^2 k_{22}^{-1}$$
$$\hat{C}_{1122} = k_{12} k_{22}^{-1},\quad \hat{C}_{1133} = k_{13} + k_{12} k_{23} k_{22}^{-1},\quad \hat{C}_{2233} = k_{23} k_{22}^{-1}$$
$$\hat{C}_{1212} = \langle C_{1212}^{-1} \rangle^{-1},\quad \hat{C}_{1313} = \langle C_{1313} \rangle,\quad \hat{C}_{2323} = \langle C_{2323}^{-1} \rangle^{-1} \tag{57}$$

where the coefficients k_{ij} can be written, alternatively, in terms of elasticity constitutive constants and engineering constants as:

$$k_{11} = \langle C_{1111} - C_{1122}^2 C_{2222}^{-1} \rangle = \langle E_1/(1-\nu_{13}\nu_{31}) \rangle$$
$$k_{22} = \langle C_{2222}^{-1} \rangle = \langle (1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13})/[E_2(1-\nu_{31}\nu_{13})] \rangle$$
$$k_{33} = \langle C_{3333} - C_{2233}^2 C_{2222}^{-1} \rangle = \langle E_3/(1-\nu_{13}\nu_{31}) \rangle$$

Table 3. Plane stress: Effective elastic constants and first eigenvalue of the homogenized material obtained from the two different limit processes

	$\hat{E}_1$	$\hat{E}_2$	$\hat{G}_{12}$	$\hat{v}_{12}$	$\hat{E}$	$\hat{\varepsilon}$	$\hat{\lambda}_1$
$\{t \to 0, \varepsilon \to 0\}$	14.000	0.885	0.550	0.340	24.775	0.161	0.6546
$\{\varepsilon \to 0, t \to 0\}$	14.007	0.984	0.550	0.369	24.729	0.153	0.6595

$$k_{12} = \langle C_{1122}C_{2222}^{-1}\rangle = \langle(v_{12}+v_{13}v_{32})/(1-v_{13}v_{31})\rangle$$
$$k_{13} = \langle C_{1133}-C_{1122}C_{2233}C_{2222}^{-1}\rangle = \langle v_{31}E_1/(1-v_{13}v_{31})\rangle$$
$$k_{23} = \langle C_{2233}C_{2222}^{-1}\rangle = \langle(v_{32}+v_{12}v_{31})/(1-v_{13}v_{31})\rangle. \quad (58)$$

The effective reduced elastic coefficients $\hat{R}_{ij}$ in eqns (57, 58) of the homogenized material can be determined from three-dimensional elasticity making use of plane deformation hypothesis. In this way, the plane stress hypothesis is used a-posteriori for the homogenized material as the limit (leading term) for the beam width approaching zero ($t \to 0$). The following elastic coefficients are obtained for plane strain

$$\hat{R}_{11} = k_{11}^{-1},\ \hat{R}_{22} = k_{22}+k_{12}^2k_{11}^{-1},\ \hat{R}_{12} = -k_{12}k_{11}^{-1},\ \hat{R}_{66} = \langle G_{12}^{-1}\rangle \quad (59)$$

and generalized plane stress, respectively:

$$\hat{R}_{11} = (k_{11}-k_{13}^2k_{33}^{-1})^{-1},\ \hat{R}_{22} = (k_{22}+k_{12}^2+k_{11}k_{23}^2k_{33}^{-1}-2k_{12}k_{13}k_{23}k_{33}^{-1})\hat{R}_{11}$$
$$\hat{R}_{12} = -(k_{12}-k_{13}k_{23}k_{33}^{-1})\hat{R}_{11},\ \hat{R}_{66} = \langle G_{12}^{-1}\rangle. \quad (60)$$

For plane strain, it can be easily verified that eqns (52) and (59) coincide. On the contrary, for generalized plane stress two different sets of effective elastic moduli $\hat{R}_{ij}$ are obtained, eqns (52, 4) and (6), i.e., the two different limit processes {first $t \to 0$, then $\varepsilon \to 0$} and {first $\varepsilon \to 0$, then $t \to 0$} do not commute.

For instance, for the bimaterial strip with S-glass epoxy-wood layers of Fig. 8, the effective elastic constants and the first eigenvalue of the homogenized material are reported in Table 3. Note that the Young moduli in the (fiber) axial direction are very close, whereas for the elastic moduli governing the behavior in the transverse direction ($\hat{E}_2$ and $\hat{v}_{12}$) the difference is close to 10 per cent.

This result is well known in homogenization theory. For instance, for the Kirchhoff-Love plate theory, Caillerie (1984, 1987) proved that the two limits for the thickness of the plate ($t \to 0$) and the relative size of the periodic structure ($\varepsilon \to 0$) do not commute. In fact, both limit processes yield the same plate equation, but with different stiffness coefficients $\hat{a}_{\alpha\beta\gamma\delta}$.

Acknowledgements—The financial support of the (Italian) Ministry of University and Scientific and Technological Research (MURST–40%) and of Human Capital Programme (contr. no. CHRX-CT93-0383-DG 12 COMA) is gratefully acknowledged. The authors are pleased to acknowledge the comments on the manuscript by Prof. C. O. Horgan.

REFERENCES

Baxter, S. C. and Horgan, C. O. (1995) End effects for anti-plane shear deformations of sandwich structures. *Journal of Elasticity* **40**, 123–164.

Bensoussan, A., Lions, J. L. and Papanicolaou, G. (1978) *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam.

Caillerie, D. (1984) Thin elastic and periodic plates. *Mathematical Methods in Applied Science* **6**, 159–191.

Caillerie, D. (1987) Non homogeneous plate theory and conduction in fibered composites. in *Homogenization Techniques for Composite Media*, eds. E. Sanchez-Palencia and A. Zaoui, pp. 1–62. Lecture Notes in Physics 272, Springer-Verlag, Berlin.

Choi, I. and Horgan, C. O. (1977) Saint-Venant's principle and end effects in anisotropic elasticity. *Journal of Applied Mechanics ASME* **44**, 424–430.

Choi, I. and Horgan, C. O. (1978) Saint-Venant end effects for plane deformation of sandwich strips. *International Journal of Solids and Structures* **14**, 187–195.

Crafter, E. C., Heise, R. M., Horgan, C. O. and Simmonds, J. G. (1993) The eigenvalues for a self-equilibrated, semi-infinite, anisotropic elastic strip. *Journal of Applied Mechanics ASME* **60**, 276–281.

Dong, S. B. and Goetschel, D. B. (1982) Edge effects in laminated composite plates. *Journal of Applied Mechanics ASME* **49**, 129–135.

Dumontet, H. (1986) Boundary layer stresses in elastic composites. In *Local Effects in the Analysis of Structures*. ed. Ladevèze P., Elsevier, Amsterdam.

Dundurs, J. (1969) Discussion. *Journal of Applied Mechanics ASME* **36**, 650–652.

Duvaut, G. (1976) Analyse fonctionelle et mécanique des milieux continue. Application à l'étude des matériaux composites élastiques à structure périodique-Homogénéisation. In *Theoretical and Applied Mechanics*, ed. W. T. Koiter, pp. 119–132. North-Holland, Amsterdam.

Folkes, M. J. and Arridge, R. G. C. (1975) The measurement of shear modulus in highly anisotropic materials: the validity of St. Venant principle. *Journal of Physics D* **8**, 1053–1064.

Gregory, R. D. (1980) The traction boundary value problem for the electrostatic semi-infinite strip; existence of solution, and completeness of the Papkovich-Fadle eigenfunction. *Journal of Elasticity* **10**, 295–327.

Gregory, R. D. (1983) A note on bi-orthogonality relations for elastic cylinders of general cross section. *Journal of Elasticity* **13**, 351–355.

Horgan, C. O. (1982) Saint-Venant end effects in composites. *Journal of Composite Materials* **16**, 411–422.

Horgan, C. O. (1989) Recent developments concerning Saint-Venant's principle; an update. *Applied Mechanics Review* **42**, 295–303.

Horgan, C. O. (1996) Recent developments concerning Saint-Venant's principle: a second update. *Applied Mechanics Review* **49**, S101–S111.

Horgan, C. O. and Knowles, J. K. (1983) Recent developments concerning Saint-Venant's principle. In *Advances in Applied Mechanics*, Wu, T. Y. and Hutchinson, J. W., vol. 23, pp. 179–269, Academic Press, New York.

Horgan, C. O. and Simmonds, J. G. (1991) Asymptotic analysis of an end-loaded transversely isotropic, elastic, semi-infinite strip weak in shear. *International Journal of Solids and Structures* **27**, 1895–1914.

Horgan, C. O. and Simmonds, J. G. (1994) Saint-Venant end effects in composite structures. *Composite Engineering* **4**, 279–286.

Kim, Y. Y. and Steele, C. R. (1990) Modifications of series expansions for general end conditions and corner singularities on the semi-infinite strip. *Journal of Applied Mechanics ASME* **57**, 581–588.

Kohn, R. V. and Vogelius, M. (1984) A new model for thin plates with rapidly varying thickness. *International Journal of Solids and Structures* **20**, 333–350.

Lekhnitskii, S. G. (1981) *Theory of Elasticity of an Anisotropic Body*. Mir Publ., Moscow.

Lewinski, T. (1991) Effective models of composite periodic plates-III. Two-dimensional approaches. *International Journal of Solids and Structures* **27**, 1185–1203.

Lin, Y. H. and Wan, F. Y. M. (1990) Bending and flexure of semi-infinite cantilevered orthotropic strips. *Computers and Structures* **35**, 349–359.

Miller, K. L. and Horgan, C. O. (1995a) Saint-Venant end effects for plane deformations of elastic composites. *Mechanics of Composite Materials and Structures* **2**, 203–214.

Miller, K. L. and Horgan, C. O. (1995b) End effects for plane deformations of an elastic anisotropic semi-infinite strip. *Journal of Elasticity* **38**, 261–316.

Oleinik, O. A., Shamaev, A. S. and Yosifian, G. A. (1992) *Mathematical Problems in Elasticity and Homogenization*. North Holland, Amsterdam.

Pagano, N. J. (1969) Exact solutions for composite laminates in cylindrical bending. *Journal of Composite Materials* **3**, 398–411.

Plevris, N. and Triantafillou, T. C. (1992) FRP-reinforced wood as structural material. *Journal of Materials in Civil Engineering ASCE* **4**, 300–317.

Reddy, J. N. (1986) *Applied Functional Analysis and Variational Methods in Engineering*. McGraw Hill, New York.

Sanchez-Palencia, E. (1980) *Non-Homogeneous Media and Vibration Theory*. Lecture Notes in Physics 127, Springer-Verlag, Berlin.

Savoia, M. and Tullini, N. (1996) Beam theory for strongly orthotropic materials. *International Journal of Solids and Structures* **33**, 2459–2484.

Timoshenko, S. P. and Goodier, J. N. (1970) *Theory of Elasticity*, 3rd ed. McGraw Hill Book Co., New York.

Ting, T. C. T. (1995) Generalized Dundurs constants for anisotropic bimaterials. *International Journal of Solids and Structures* **32**, 483–500.

Tullini, N. and Savoia, M. (1995) Logarithmic stress singularities at the clamped-free corners of a cantilever orthotropic beam under flexure. *Composite Structures* **32**, 659–666.

Wang, M. Z., Ting, T. C. T. and Yan, G. (1993) The anisotropic elastic semi-infinite strip. *Quarterly of Applied Mathematics* **51**, 283–297.

Wijeyewickrema, A. C. (1995) Decay of stresses induced by self-equilibrated end loads in a multilayered composite. *International Journal of Solids and Structures* **32**, 515–523.

Wijeyewickrema, A. C., Horgan, C. O. and Dundurs, J. (1996) Further analysis of end effects for plane deformations of sandwich strips. *International Journal of Solids and Structures* **33**, 4327–4336.